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# WEIGHTED HARDY AND PÓLYA–KNOPP INEQUALITIES WITH VARIABLE LIMITS

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*Abstract.* A new scale of characterizations for the weighted Hardy inequality with variable limits is proved for the case $1 < p \leqslant q < \infty$. A corresponding scale of characterizations for the (limit) weighted Pólya-Knopp inequality is also derived and discussed.

## 1. Introduction

Let $a = a(x)$, $b = b(x)$ be strictly increasing differentiable functions on $[0, \infty]$ satisfying

$$\begin{aligned} &a(0) = b(0) = 0, \\ &a(x) < b(x) \qquad \text{for } 0 < x < \infty, \\ &a(\infty) = b(\infty) = \infty. \end{aligned} \tag{1.1}$$

In [1] it was proved that for the case $1 < p \leqslant q < \infty$ the inequality

$$\left( \int_0^\infty \left( \int_{a(x)}^{b(x)} f(t)dt \right)^q u(x)dx \right)^{\frac{1}{q}} \leqslant C \left( \int_0^\infty f^p(x)v(x)dx \right)^{\frac{1}{p}} \tag{1.2}$$

holds for some finite constant $C$ and for all positive measurable functions $f$ if and only if

$$A = \sup \left( \int_t^x u(z)dz \right)^{\frac{1}{q}} \left( \int_{a(x)}^{b(t)} v^{1-p'}(z)dz \right)^{\frac{1}{p'}} < \infty, \tag{1.3}$$

where supremum is taken over all $x$ and $t$ such that

$$0 < t < x < \infty \qquad \text{and} \qquad a(x) < b(t). \tag{1.4}$$

Moreover, if $C$ is the best constant for which (1.2) holds, then

$$A \leqslant C \leqslant 2\left(1 + \frac{q}{p'}\right)^{\frac{1}{q}} \left(1 + \frac{p'}{q}\right)^{\frac{1}{p'}} A, \tag{1.5}$$

where $p' = p/(p-1)$.

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In this paper we prove that the condition (1.3) is not unique for characterizing (1.2). More exactly, inspired by results in [5] we will prove that there is a whole scale of conditions for characterizing (1.2) (see Theorem 2.1). In fact, (1.3) may be regarded as just an endpoint case of this scale (see Remark 2.2). We note that (1.2) describes the (continuous) mapping properties of the (arithmetic mean) Hardy type operator

$$(Hf)(x) = \frac{1}{b(x)-a(x)} \int_{a(x)}^{b(x)} f(t)dt.$$

In [4] the corresponding geometric mean operator with variable limits was introduced as

$$(Gf)(x) = \exp\left( \frac{1}{b(x)-a(x)} \int_{a(x)}^{b(x)} \ln f(t)dt \right),$$

and the Pólya-Knopp type inequality

$$\left( \int_0^\infty (Gf)^q(x)u(x)dx \right)^{\frac{1}{q}} \leqslant C \left( \int_0^\infty f^p(x)v(x)dx \right)^{\frac{1}{p}} \tag{1.6}$$

was characterized for the case $0 < p \leqslant q < \infty$ with the following weight characterization:

$$A_{PP} = \sup_{t>0} \left( \int_{\sigma^{-1}(a(x))}^{\sigma^{-1}(b(x))} w(x)dx \right)^{\frac{1}{q}} (b(t)-a(t))^{-\frac{1}{p}} < \infty, \tag{1.7}$$

where

$$\sigma(t) = \frac{a(t)+b(t)}{2},$$

and

$$w(x) = \exp\left( \frac{1}{b(x)-a(x)} \int_{a(x)}^{b(x)} \ln \frac{1}{v(t)} dt \right)^{\frac{q}{p}} u(x). \tag{1.8}$$

In this paper we also prove that the condition (1.7) is not unique for characterizing (1.6). In fact, we will prove that also in this case there exists a scale of conditions for characterizing (1.6) (see Theorem 3.1). The method of proof is completely different (and simpler) than that in [4].

## 2. A scale of weight characterizations for the Hardy inequality with variable limits

Our main result in this Section reads:

THEOREM 2.1. *Let $1 < p \leqslant q < \infty$. Then the inequality (1.2) holds for some finite constant $C$ and for all positive measurable functions $f$ if and only if for any $s \in (1,p)$*

$$B(s) := \sup \left( \int_t^x u(z) \left( \int_{a(z)}^{b(z)} v^{1-p'}(\eta) d\eta \right)^{\frac{q(p-s)}{p}} dz \right)^{\frac{1}{q}} \left( \int_{a(x)}^{b(t)} v^{1-p'}(z) dz \right)^{\frac{s-1}{p}} < \infty, \quad (2.1)$$

*where supremum is taken over all* $x$ *and* $t$ *such that* $(1.4)$ *holds. Moreover, if* $C$ *is the best constant in* $(1.2)$*, then*

$$\sup_{1<s<p} \left( \frac{\left(\frac{p}{p-s}\right)^p}{\left(\frac{p}{p-s}\right)^p + \frac{1}{s-1}} \right)^{\frac{1}{p}} B(s) \leqslant C \leqslant 2 \inf_{1<s<p} \left( \frac{p-1}{p-s} \right)^{\frac{1}{p'}} B(s). \quad (2.2)$$

REMARK 2.2. Note that if $s \to p$, then $B(s) \to A$ as defined in $(1.3)$ and also the lower bound in $(2.2)$ converges to the lower bound in $(1.5)$.

For the proof of Theorem 2.1 we need the following result of A. Wedestig [5]:

LEMMA 2.3. *Let* $1 < p \leqslant q < \infty$*. Then for* $0 \leqslant a < b \leqslant \infty$ *the inequality*

$$\left( \int_a^b \left( \int_a^x f(t) dt \right)^q u(x) dx \right)^{\frac{1}{q}} \leqslant C \left( \int_a^b f^p(x) v(x) dx \right)^{\frac{1}{p}} \quad (2.3)$$

*holds for some finite constant* $C$ *and for all measurable functions* $f \geqslant 0$ *if and only if for any* $s \in (1,p)$

$$A_W(s) = \sup_{a \leqslant t \leqslant b} \left( \int_t^b u(x) \left( \int_a^x v^{1-p'}(y) dy \right)^{\frac{q(p-s)}{p}} dx \right)^{\frac{1}{q}} \left( \int_a^t v^{1-p'}(y) dy \right)^{\frac{s-1}{p}} < \infty. \quad (2.4)$$

*Moreover, if* $C$ *is the best constant in* $(2.3)$*, then*

$$\sup_{1<s<p} \left( \frac{\left(\frac{p}{p-s}\right)^p}{\left(\frac{p}{p-s}\right)^p + \frac{1}{s-1}} \right)^{\frac{1}{p}} A_W(s) \leqslant C \leqslant \inf_{1<s<p} \left( \frac{p-1}{p-s} \right)^{\frac{1}{p'}} A_W(s). \quad (2.5)$$

By using a standard duality argument (see e.g. [3]) we also have:

LEMMA 2.4. *Let* $1 < p \leqslant q < \infty$*. Then for* $0 \leqslant a < b \leqslant \infty$ *the inequality*

$$\left( \int_a^b \left( \int_x^b f(t) dt \right)^q u(x) dx \right)^{\frac{1}{q}} \leqslant C \left( \int_a^b f^p(x) v(x) dx \right)^{\frac{1}{p}} \quad (2.6)$$

*holds for some constant* $C$ *and for all measurable positive functions* $f$ *if and only if for any* $s \in (1,p)$

$$\widetilde{A}_W(s) = \sup_{a \leqslant t \leqslant b} \left( \int_a^t u(x) \left( \int_x^b v^{1-p'}(y)dy \right)^{\frac{q(p-s)}{p}} dx \right)^{\frac{1}{q}} \left( \int_t^b v^{1-p'}(y)dy \right)^{\frac{s-1}{p}} < \infty. \quad (2.7)$$

*Furthermore, if* $C$ *is the best constant in* (2.6)*, then*

$$\sup_{1<s<p} \left( \frac{\left(\frac{p}{p-s}\right)^p}{\left(\frac{p}{p-s}\right)^p + \frac{1}{s-1}} \right)^{\frac{1}{p}} \widetilde{A}_W(s) \leqslant C \leqslant \inf_{1<s<p} \left( \frac{p-1}{p-s} \right)^{\frac{1}{p'}} \widetilde{A}_W(s). \quad (2.8)$$

*Proof of Theorem* 2.1.

Necessity: Put $V(t) = \int_0^t v^{1-p'}(\eta)d\eta$ and suppose that (1.2) holds. Consider the test function

$$f(y) = \frac{p}{p-s} \left(V(b(t)) - V(a(z))\right)^{-\frac{s}{p}} v^{1-p'}(y)\chi_{(a(z),b(t))}(y) + \left(V(y) - V(a(z))\right)^{-\frac{s}{p}} v^{1-p'}(y)\chi_{(b(t),b(z))}(y).$$

Here $t$ and $x$ are fix numbers, $0 < t < x < \infty$, such that $a(x) < b(t)$. It yields that if $t < z \leqslant x$, then $a(z) \leqslant a(x) < b(t) < b(z) \leqslant b(x)$. The right hand side integral in (1.2) can be estimated as follows:

$$\begin{aligned}
\left( \int_0^\infty f^p(x)dx \right)^{\frac{1}{p}} &= \left( \int_{a(z)}^{b(t)} \left(\frac{p}{p-s}\right)^p \left(V(b(t)) - V(a(z))\right)^{-s} v^{1-p'}(y)dy \right. \\
&\qquad \left. + \int_{b(t)}^{b(z)} \left(V(y) - V(a(z))\right)^{-s} v^{1-p'}(y)dy \right)^{\frac{1}{p}} \\
&\leqslant \left( \left(\frac{p}{p-s}\right)^p \left(V(b(t))-V(a(z))\right)^{1-s} + \frac{1}{s-1} \left(V(b(t))-V(a(z))\right)^{1-s} \right)^{\frac{1}{p}} \qquad (2.9) \\
&= \left( \left(\frac{p}{p-s}\right)^p + \frac{1}{s-1} \right)^{\frac{1}{p}} \left(V(b(t)) - V(a(z))\right)^{\frac{1-s}{p}} \\
&\leqslant \left( \left(\frac{p}{p-s}\right)^p + \frac{1}{s-1} \right)^{\frac{1}{p}} \left(V(b(t)) - V(a(x))\right)^{\frac{1-s}{p}}.
\end{aligned}$$

Moreover, for the left hand side in (1.2) we have

$$
\begin{aligned}
&\left(\int_0^{\infty}\left(\int_{a(z)}^{b(z)} f(y)dy\right)^q u(z)dz\right)^{\frac{1}{q}} \geqslant \left(\int_t^{x}\left(\int_{a(z)}^{b(z)} f(y)dy\right)^q u(z)dz\right)^{\frac{1}{q}}\\
&=\left(\int_t^x\left(\int_{a(z)}^{b(t)} \frac{p}{p-s}(V(b(t))-V(a(z)))^{-\frac{s}{p}}v^{1-p'}(y)dy\right.\right.\\
&\qquad\left.\left.+\int_{b(t)}^{b(z)}(V(y)-V(a(z)))^{-\frac{s}{p}}v^{1-p'}(y)dy\right)^q u(z)dz\right)^{\frac{1}{q}} \qquad (2.10)\\
&=\left(\int_t^x\left(\frac{p}{p-s}(V(b(t))-V(a(z)))^{\frac{p-s}{p}}+\frac{p}{p-s}(V(b(z))-V(a(z)))^{\frac{p-s}{p}}\right.\right.\\
&\qquad\left.\left.-\frac{p}{p-s}(V(b(t))-V(a(z)))^{\frac{p-s}{p}}\right)^q u(z)dz\right)^{\frac{1}{q}}\\
&=\frac{p}{p-s}\left(\int_t^x (V(b(z))-V(a(z)))^{\frac{q(p-s)}{p}}u(z)dz\right)^{\frac{1}{q}}.
\end{aligned}
$$

Hence, by (1.2), (2.9) and (2.10),

$$
\frac{\frac{p}{p-s}}{\left(\left(\frac{p}{p-s}\right)^p+\frac{1}{s-1}\right)^{\frac{1}{p}}}\left(\int_t^x u(z)(V(b(z))-V(a(z)))^{\frac{q(p-s)}{p}}dz\right)^{\frac{1}{q}}(V(b(t))-V(a(x)))^{\frac{s-1}{p}} \leqslant C.
$$

We conclude that (2.1) and, by taking supremum, the left hand side inequality in (2.2) holds.

Sufficiency: Assume that (2.1) holds. Here we first use some arguments from [1] (see also [3, p. 127]) and define $a=a(x)$, $b=b(x)$ to be strictly increasing differentiable functions on $[0,\infty]$ satisfying the conditions (1.1) and, consequently, the inverse functions $a^{-1}$ and $b^{-1}$ exist and are strictly increasing and differentiable, too. We define a sequence $\{m_k\}_{k\in\mathbb{Z}}$ as follows: for fixed $m>0$ define $m_0=m$ and

$$
\begin{aligned}
m_{k+1}&=a^{-1}(b(m_k)), \text{ if } k\geqslant 0,\\
m_k&=b^{-1}(a(m_{k+1})), \text{ if } k\leqslant 0.
\end{aligned} \qquad (2.11)
$$

Thus, we have

$$
a(m_{k+1})=b(m_k) \quad \text{for all } k\in\mathbb{Z}. \qquad (2.12)
$$

Moreover, we define the weight functions $u_a$ and $u_b$ by

$$
\begin{aligned}
u_a(y)&=u(a^{-1}(y))(a^{-1})'(y),\\
u_b(y)&=u(b^{-1}(y))(b^{-1})'(y),
\end{aligned} \qquad (2.13)
$$

and $a_k = a(m_k)$, $b_k = b(m_k)$, $k \in \mathbb{Z}$ (note, that $(0,\infty) = \cup[a_k, b_k]$). It follows that the left hand side of $(1.2)$ (with $f$ replaced by $f v^{1-p'}$) is less than or equal to

$$\left(\sum_{k\in\mathbb{Z}} \int_{a_k}^{b_k} \left(\int_y^{b_k} f(t) v^{1-p'}(t)dt\right)^q u_a(y)dy\right)^{\frac{1}{q}} + \left(\sum_{k\in\mathbb{Z}} \int_{a_{k+1}}^{b_{k+1}} \left(\int_{a_{k+1}}^{y} f(t) v^{1-p'}(t)dt\right)^q u_b(y)dy\right)^{\frac{1}{q}} = I_1 + I_2. \tag{2.14}$$

For details see the book [3, p. 133].

Fix $t > 0$ and let $t < z < x$. Write $y = a(x)$ in $(2.1)$ and make the variable transformation $a(z) = r$. Then

$$\begin{aligned}
B(s) &\geqslant \left(\int_t^{a^{-1}(y)} u(z) \left(\int_{a(z)}^{b(z)} v^{1-p'}(\eta)d\eta\right)^{\frac{q(p-s)}{p}} dz\right)^{\frac{1}{q}} \left(\int_y^{b(t)} v^{1-p'}(z)dz\right)^{\frac{s-1}{p}} \\
&= \left(\int_{a(t)}^{y} \left(\int_r^{b(a^{-1}(r))} v^{1-p'}(\eta)d\eta\right)^{\frac{q(p-s)}{p}} u(a^{-1}(r))(a^{-1})'(r)dr\right)^{\frac{1}{q}} \left(\int_y^{b(t)} v^{1-p'}(z)dz\right)^{\frac{s-1}{p}} \\
&\geqslant \left(\int_{a(t)}^{y} \left(\int_r^{b(t)} v^{1-p'}(\eta)d\eta\right)^{\frac{q(p-s)}{p}} u_a(r)dr\right)^{\frac{1}{q}} \left(\int_y^{b(t)} v^{1-p'}(z)dz\right)^{\frac{s-1}{p}},
\end{aligned}$$

where $u_a$ is defined by $(2.13)$. In the last estimate we have used that $b(t) < b(a^{-1}(r))$ (which holds because $t < z < a^{-1}(r)$). We conclude that

$$\left(\int_a^y \left(\int_r^b v^{1-p'}(\eta)d\eta\right)^{\frac{q(p-s)}{p}} u_a(r)dr\right)^{\frac{1}{q}} \left(\int_y^b v^{1-p'}(z)dz\right)^{\frac{s-1}{p}} \leqslant B(s) < \infty \tag{2.15}$$

for all $(a,b) = (a(t), b(t))$ (and, thus, in particular for all $(a,b) = (a_k, b_k)$, $k \in \mathbb{Z}$). Hence, (2.7) is satisfied and we can use Lemma 2.4 repeatedly (with a uniform bound of the best constant) and obtain

$$
\begin{aligned}
I_1 &\leqslant \left( \sum_{k\in\mathbb{Z}} C^q \left( \int_{a_k}^{b_k} \left( f(y) v^{1-p'}(y) \right)^p v(y)dy \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \\
&\leqslant C \left( \sum_{k\in\mathbb{Z}} \int_{a_k}^{b_k} f^p(y) v^{1-p'}(y)dy \right)^{\frac{1}{p}} = C \left( \int_0^\infty f^p(y) v^{1-p'}(y)dy \right)^{\frac{1}{p}},
\end{aligned} \tag{2.16}
$$

where, by $(2.8)$ and $(2.15)$,

$$
C \leqslant \inf_{1<s<p} \left( \frac{p-1}{p-s} \right)^{\frac{1}{p'}} \widetilde{A}_W(s) \leqslant \inf_{1<s<p} \left( \frac{p-1}{p-s} \right)^{\frac{1}{p'}} B(s). \tag{2.17}
$$

Next, we let $x$ be fixed, write $y = b(t)$ in $(2.1)$ and make the variable transformation $b(z) = r$. Similar as above we find that

$$
\left( \int_y^b \left( \int_a^r v^{1-p'}(\eta)d\eta \right)^{\frac{q(p-s)}{p}} u_b(r)dr \right)^{\frac{1}{q}} \left( \int_a^y v^{1-p'}(z)dz \right)^{\frac{s-1}{p}} \leqslant B(s) < \infty \tag{2.18}
$$

for all $(a,b) = (a(t), b(t))$ and where $u_b$ is defined by $(2.13)$. Thus, $(2.4)$ holds and we can use Lemma 2.3 to find that

$$
\begin{aligned}
I_2 &\leqslant \left( \sum_{k\in\mathbb{Z}} C^q \left( \int_{a_{k+1}}^{b_{k+1}} \left( f(y) v^{1-p'}(y) \right)^p v(y)dy \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \\
&\leqslant C \left( \sum_{k\in\mathbb{Z}} \int_{a_{k+1}}^{b_{k+1}} f^p(y) v^{1-p'}(y)dy \right)^{\frac{1}{p}} = C \left( \int_0^\infty f^p(y) v^{1-p'}(y)dy \right)^{\frac{1}{p}},
\end{aligned} \tag{2.19}
$$

where, by $(2.5)$ and $(2.18)$,

$$
C \leqslant \inf_{1<s<p} \left( \frac{p-1}{p-s} \right)^{\frac{1}{p'}} A_W(s) \leqslant \inf_{1<s<p} \left( \frac{p-1}{p-s} \right)^{\frac{1}{p'}} B(s). \tag{2.20}
$$

By now combining $(2.14)$ with $(2.16)$ and $(2.19)$ we find that the left hand side of $(2.1)$ can be estimated by

$$
C \left( \int_0^\infty f^p(y) v(y)dy \right)^{\frac{1}{p}}
$$

(here we have replaced $f(y)v^{1-p'}(y)$ by $f(y)$ again), where by $(2.17)$ and $(2.20)$ the best constant can be estimated as stated in the right hand side inequality of $(2.2)$. The proof is complete.

## 3. A scale of weight characterizations for the Pólya-Knopp inequality with variable limits

Our main result in this Section reads:

THEOREM 3.1. *Let* $0 < p \leqslant q < \infty$. *Then the inequality* (1.6) *holds for all positive measurable functions* $f$ *if and only if for any* $s > 1$

$$D(s) := \sup \left( \int_t^x w(z) \left(b(z) - a(z)\right)^{\frac{-qs}{p}} dz \right)^{\frac{1}{q}} \left(b(t) - a(x)\right)^{\frac{s-1}{p}} < \infty, \tag{3.1}$$

*where the supremum is taken over all* $x$ *and* $t$ *such that* (1.4) *holds and* $w(z)$ *is defined by* (1.8)*. Moreover, if* $C$ *is the best constant in* (1.6)*, then*

$$\begin{aligned} \sup_{s>1} \left( \frac{e^s(s-1)}{e^s(s-1)+1} \right)^{\frac{1}{p}} D(s) &\leqslant C \\ &\leqslant \inf_{s>1} 2^{\frac{s}{p}} \left( \frac{p+q(s-1)}{p} \right)^{\frac{1}{q}} \left( \frac{p+q(s-1)}{q(s-1)} \right)^{\frac{s-1}{p}} D(s). \end{aligned} \tag{3.2}$$

*Proof.* Sufficiency: Assume that (3.1) holds. If we define $w(z)$ as in (1.8) and replace $f^p(x)$ by $g(x)/v(x)$ we see that (1.6) is equivalent to

$$\left( \int_0^\infty \left( \exp\left( \frac{1}{b(x) - a(x)} \int_{a(x)}^{b(x)} \ln g(t) dt \right) \right)^{\frac{q}{p}} w(x) dx \right)^{\frac{1}{q}} \leqslant C \left( \int_0^\infty g(x) dx \right)^{\frac{1}{p}}. \tag{3.3}$$

Now we let $g(x) = f^s(x)$. Then we have that (3.3) and, thus, (1.6) is equivalent to

$$\left( \int_0^\infty \left( \exp\left( \frac{1}{b(x) - a(x)} \int_{a(x)}^{b(x)} \ln f(t) dt \right) \right)^{\frac{qs}{p}} w(x) dx \right)^{\frac{1}{q}} \leqslant C \left( \int_0^\infty f^s(x) dx \right)^{\frac{1}{p}}. \tag{3.4}$$

Moreover, by Jensen's inequality it follows that

$$\begin{aligned} &\left( \int_0^\infty \left( \exp\left( \frac{1}{b(x) - a(x)} \int_{a(x)}^{b(x)} \ln f(t) dt \right) \right)^{\frac{qs}{p}} w(x) dx \right)^{\frac{1}{q}} \\ &\qquad \leqslant \left( \int_0^\infty \left( \int_{a(x)}^{b(x)} f(t) dt \right)^{\frac{qs}{p}} w(x) (b(x) - a(x))^{-\frac{qs}{p}} dx \right)^{\frac{1}{q}}. \end{aligned} \tag{3.5}$$

We only need to consider the inequality

$$\left(\int_0^\infty \left(\int_{a(x)}^{b(x)} f(t)dt\right)^{\frac{qs}{p}} w(x)(b(x)-a(x))^{-\frac{qs}{p}}dx\right)^{\frac{1}{q}} \leqslant C\left(\int_0^\infty f^s(x)dx\right)^{\frac{1}{p}}. \tag{3.6}$$

In order to be able to compare with (1.2) we note that (3.6) can equivalently be written as

$$\left(\int_0^\infty \left(\int_{a(x)}^{b(x)} f(t)dt\right)^{\frac{qs}{p}} w(x)(b(x)-a(x))^{-\frac{qs}{p}}dx\right)^{\frac{p}{qs}} \leqslant C_0\left(\int_0^\infty f^s(x)dx\right)^{\frac{1}{s}}, \tag{3.7}$$

where $C_0 = C^{\frac{p}{s}}$. Note that (3.7) is just (1.2) with $v(x) \equiv 1$, $u(x) = w(x)(b(x)-a(x))^{-\frac{qs}{p}}$, $p$ replaced by $s$ and $q$ replaced by $\frac{qs}{p}$. Hence, according to [1], it holds (see (1.3)) if

$$A = A(s) = \sup\left(\int_t^x w(z)(b(z)-a(z))^{-\frac{qs}{p}}dz\right)^{\frac{p}{qs}} (b(t)-a(x))^{\frac{s-1}{s}} < \infty,$$

where supremum is taken over all $x$ and $t$ such that (1.4) holds. In fact, this condition holds in view of the assumption (3.1) (note that $A(s) = D(s)^{\frac{p}{s}}$ ). Therefore, (3.6) holds and, according to (3.5), it follows that (3.6) and, thus, (1.6) holds. Moreover, if $C_0$ is the best possible constant in (3.7), then (see (1.5))

$$C_0 \leqslant \inf_{s>1} 2\left(\frac{p+q(s-1)}{p}\right)^{\frac{p}{qs}} \left(\frac{p+q(s-1)}{q(s-1)}\right)^{\frac{s-1}{s}} A(s).$$

We conclude that (3.6) and, thus, (1.6) holds and for the best constant $C$ in (1.6) it yields that

$$C \leqslant \inf_{s>1} 2^{\frac{s}{p}} \left(\frac{p+q(s-1)}{p}\right)^{\frac{1}{q}} \left(\frac{p+q(s-1)}{q(s-1)}\right)^{\frac{s-1}{p}} D(s). \tag{3.8}$$

Necessity: Assume that (1.6) and, thus, (3.3) holds with some finite constant $C$. Let $t$ and $x$ be fix numbers, such that $0 < t < x < \infty$ and $a(x) < b(t)$. For any $z \in (t,x)$ it yields that $a(z) \leqslant a(x) < b(t) < b(z) \leqslant b(x)$. To (3.3) we now apply the following test function:

$$g(y) = (b(t)-a(z))^{-s}\chi_{(a(z),b(t))}(y) + e^{-s}(y-a(z))^{-s}\chi_{(b(t),b(z))}(y).$$

Then the right hand side of (3.3) can be estimated as follows:

$$
\begin{aligned}
\left(\int_0^\infty g(y)dy\right)^{\frac{1}{p}} &= \left(\int_{a(z)}^{b(t)} (b(t)-a(z))^{-s}dy + e^{-s}\int_{b(t)}^{b(z)} (y-a(z))^{-s}dy\right)^{\frac{1}{p}} \\
&\leqslant \left((b(t)-a(z))^{1-s} + e^{-s}\frac{1}{s-1}(b(t)-a(z))^{1-s}\right)^{\frac{1}{p}} \\
&\leqslant \left(1+e^{-s}\frac{1}{s-1}\right)^{\frac{1}{p}} (b(t)-a(x))^{\frac{1-s}{p}}.
\end{aligned} \tag{3.9}
$$

For the left hand side in (3.3) we have

$$
\begin{aligned}
I_L := &\left(\int_0^\infty \left(\exp\left(\frac{1}{b(z)-a(z)}\int_{a(z)}^{b(z)} \ln g(y)dy\right)^{\frac{q}{p}}\right) w(z)dz\right)^{\frac{1}{q}} \\
\geqslant &\left(\int_t^x \left(\exp\left(\frac{1}{b(z)-a(z)}\int_{a(z)}^{b(z)} \ln g(y)dy\right)^{\frac{q}{p}}\right) w(z)dz\right)^{\frac{1}{q}} \\
= &\left(\int_t^x \left(\exp\left(\frac{1}{b(z)-a(z)}\left[\int_{a(z)}^{b(t)} \ln(b(t)-a(z))^{-s}dy\right.\right.\right.\right. \\
&\left.\left.\left.\left. + \int_{b(t)}^{b(z)} \ln e^{-s}(y-a(z))^{-s}dy\right]\right)^{\frac{q}{p}}\right) w(z)dz\right)^{\frac{1}{q}} \\
=: &\left(\int_t^x \left(\exp\left(\frac{1}{b(z)-a(z)}[I+II]\right)^{\frac{q}{p}}\right) w(z)dz\right)^{\frac{1}{q}},
\end{aligned}
$$

where

$$I = -s(b(t)-a(z))\ln(b(t)-a(z)),$$

and

$$
\begin{aligned}
II &= -s\int_{b(t)}^{b(z)} dy - s\int_{b(t)}^{b(z)} \ln(y-a(z))dy \\
&= -s(b(z)-a(z))\ln(b(z)-a(z)) + s(b(t)-a(z))\ln(b(t)-a(z)).
\end{aligned}
$$

Summing up we have

$$I + II = -s(b(z)-a(z))\ln(b(z)-a(z)),$$

and we conclude that

$$
\begin{aligned}
I_L &\geqslant \int_t^x \left(\exp\left(-s\cdot\ln(b(z)-a(z))\right)\right)^{\frac{q}{p}} w(z)dz \\
&= \left(\int_t^x (b(z)-a(z))^{-\frac{sq}{p}} w(z)dz\right)^{\frac{1}{q}}.
\end{aligned} \tag{3.10}
$$

Thus, by combining (3.8) and (3.9) we find that

$$C \geqslant \left(1 + e^{-s}\frac{1}{s-1}\right)^{-\frac{1}{p}} D(s) = \left(\frac{e^s(s-1)}{e^s(s-1)+1}\right)^{\frac{1}{p}} D(s). \tag{3.11}$$

Hence, (3.1) holds and, moreover, by combining (3.8) and (3.11) we see that also (3.2) holds so the proof is complete.

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